A category for the adjoint representation

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1 Introduction

The adjoint representation of a simple Lie algebra $\mathfrak{g}$ admits a deformation into an irreducible representation R of the quantum group $U_q(\mathfrak{g})$. In this paper for a simply-laced $\mathfrak{g}$ we realize R as the Grothendieck group of a particular abelian category $\mathcal{C}$. There are exact functors from $\mathcal{C}$ to $\mathcal{C}$ which on the Grothendieck group act as the quantum group generators E_α, F_α, where α varies over simple roots. Various relations in the quantum group between products of E_α and F_α become functor isomorphisms.

The adjoint representation R has a weight space decomposition as the direct sum of 1-dimensional vector spaces, one for each root of $\mathfrak{g}$, and the Cartan subalgebra. Mirroring this, we define $\mathcal{C}$ as the direct sum of copies of the category of graded vector spaces and the category of graded modules over the algebra $A(\Gamma)$, naturally associated to the Dynkin diagram Γ of $\mathfrak{g}$. Change each edge of Γ into a pair of oriented edges, form the path algebra of this oriented graph, and quotient it out by the ideal generated by certain linear combinations of length 2 paths. $A(\Gamma)$ is the resulting quotient algebra, and we name it the *zigzag algebra* of Γ. The Grothendieck group of the category of $A(\Gamma)$-modules is naturally identified with the weight lattice in the Cartan subalgebra of $\mathfrak{g}$.

We introduce functors $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ lifting the generators E_α and F_α of $U_q(\mathfrak{g})$ and check that defining relations in the quantum group become isomorphisms of functors. We proceed to explore various properties of our categorification of the quantum group action on R. Among them is the adjointness of functors $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$, existence of several dualities in $\mathcal{C}$ and a braid group action in the derived category of $A(\Gamma)$-modules.

We expect that not just the adjoint but any finite-dimensional irreducible representation L of the quantum group $U_q(\mathfrak{g})$, for a simple simply-laced Lie algebra $\mathfrak{g}$, admits a canonical realization as the Grothendieck group of an abelian category $\mathcal{C}(L)$. In this realization the Kashiwara-Lusztig basis in L should become the basis of indecomposable projective objects, the quantum group should act by exact functors and there should be a braid group action in the derived category of $\mathcal{C}(L)$. In short, all structures of the category $\mathcal{C}$ that we describe in this paper should also be present in categories $\mathcal{C}(L)$. Categories $\mathcal{C}(L)$ will be very close relatives of categories of coherent sheaves on Nakajima quiver varieties [Na] and categories of modules over cyclotomic Hecke algebras [A]. The work of Ariki [A], among other things, contains a categorification of all irreducible finite-dimensional representations of $\mathfrak{sl}_n$. His categories are made of blocks of the categories of modules over cyclotomic Hecke algebras for generic q. Ariki's goals, which include a proof and generalizations of the Lascoux-Leclerc-Thibon conjecture [LLT], are quite different from ours. In particular, it has not been checked whether various fine structures of the category $\mathcal{C}$, described in Section 4 of our paper and expected to hold in categories $\mathcal{C}(L)$, are present in Ariki's categories.

This work is intended to provide a simple model example of a "perfect" categorification, with all structures visible in the representation L lifted to its categorification $\mathcal{C}(L)$. Another model example, a categorification of irreducible $U_q(\mathfrak{sl}_2)$ representations, will be treated in [Kh].

Our second goal is to draw the reader's attention to the zigzag algebra $A(\Gamma)$ of a graph Γ. Zigzag algebras have a variety of nice features, which we discuss in Sections 5 and 6:

(i) $A(\Gamma)$ is a trivial extension algebra and has a nondegenerate symmetric trace form;

(ii) if Γ is a finite Dynkin diagram, then $A(\Gamma)$ has finite type and there is a bijection between indecomposable representations of $A(\Gamma)$ and roots of $\mathfrak{g}$;

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(iii) if Γ is bipartite, the quadratic dual of the zigzag algebra is the preprojective algebra of Γ, for a sink-source orientation of Γ;

(iv) Any multiplicity one Brauer tree algebra is derived equivalent to a zigzag algebra;

(v) if Γ is a bipartite affine Dynkin diagram, $A(\Gamma)$ is Morita equivalent to the cross-product algebra $\Lambda(\mathbb{C}^2, G)$, where G is the finite subgroup of $SU(2)$ associated to Γ via the McKay correspondence, and $\Lambda(\mathbb{C}^2, G)$ is the cross-product of the group algebra of G and the exterior algebra on 2 generators.

Section 6 expands on (v) to explain the role played by the zigzag algebras in the McKay correspondence. This section can be viewed as a comment on a recent work of Kapranov and Vasserot [KV], where categories of coherent sheaves on resolutions of simple surface singularities are related to categories of modules over the cross-product of $\mathbb{C}[x,y]$ and the group algebra of G, the latter cross-product being Koszul dual to $\Lambda(\mathbb{C}^2, G)$.

We conclude the paper with Section 7, where we compile a surprisingly long and diverse list of other appearances of zigzag algebras in the representation theory and geometry.

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2 The adjoint representation of a simply-laced quantum group

2.1 Quantum groups

Let $\mathfrak{g}$ be a complex simple simply-laced Lie algebra, Φ the root system of $\mathfrak{g}$ and Π a set of simple roots. The Weyl group W of $\mathfrak{g}$ acts on the real vector space $\mathbb{R}\Phi$ and there is a unique W-invariant bilinear form on $\mathbb{R}\Phi$ such that $(\alpha,\alpha) = 2$ for any root $\alpha \in \Phi$.

Let $\mathbb{Q}(q)$ be the field of polynomial functions with rational coefficients in an indeterminate q.

The quantum group $U = U_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-algebra with generators $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1}$ for $\alpha \in \Pi$ and relations

$$\begin{aligned}
&K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha,\\
&K_\alpha K_\beta = K_\beta K_\alpha,\\
&K_\alpha E_\beta = q^{(\alpha,\beta)} E_\beta K_\alpha,\\
&K_\alpha F_\beta = q^{-(\alpha,\beta)} F_\beta K_\alpha,\\
&E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}},\\
&E_\alpha E_\beta = E_\beta E_\alpha \text{ for } (\alpha,\beta) = 0,\\
&F_\alpha F_\beta = F_\beta F_\alpha \text{ for } (\alpha,\beta) = 0,\\
&E_\alpha^2 E_\beta - (q+q^{-1}) E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 = 0 \text{ for } (\alpha,\beta) = -1,\\
&F_\alpha^2 F_\beta - (q+q^{-1}) F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 = 0 \text{ for } (\alpha,\beta) = -1.
\end{aligned} \tag{1}$$

Let $^-$ be the $\mathbb{Q}$-linear involution of $\mathbb{Q}(q)$ which changes q into q^{-1}.

U has an antiautomorphism $\tau : U \to U^{\mathrm{op}}$ described by

$$\begin{aligned}
&\tau(E_\alpha) = qF_\alpha K_\alpha^{-1},\ \tau(F_\alpha) = qE_\alpha K_\alpha,\ \tau(K_\alpha) = K_\alpha^{-1},\\
&\tau(fx) = \overline{f}\tau(x), \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in U,\\
&\tau(xy) = \tau(y)\tau(x), \quad \text{for } x, y \in U.
\end{aligned} \tag{2}$$

Let ψ be the $\mathbb{Q}$-algebra involution of U defined by

$$\begin{aligned}
&\psi(E_\alpha) = E_\alpha,\ \psi(F_\alpha) = F_\alpha,\ \psi(K_\alpha) = K_\alpha^{-1},\\
&\psi(fx) = \overline{f}x \quad \text{for} \quad f \in \mathbb{Q}(q) \quad \text{and} \quad x \in U.
\end{aligned} \tag{3}$$

Let ω be the $\mathbb{Q}(q)$-linear involution of U given by

$$\omega(E_\alpha) = F_\alpha,\ \omega(F_\alpha) = E_\alpha,\ \omega(K_\alpha) = K_\alpha^{-1} \tag{4}$$

These three automorphisms and antiautomorphisms τ, ψ, ω satisfy the following relations

$$\psi\omega = \omega\psi,\ \tau\omega = \omega\tau,\ \tau\psi\tau = \psi. \tag{5}$$

Given a $\mathbb{Q}(q)$-vector space V, a $\mathbb{Q}$-bilinear form $V \times V \to \mathbb{Q}(q)$ is called semilinear if it is $\mathbb{Q}(q)$-antilinear in the first variable and $\mathbb{Q}(q)$-linear in the second, i.e.

$$< fx, y >= \overline{f} < x, y >,\ < x, fy >= f < x, y > \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x, y \in U. \tag{6}$$

2.2 The adjoint representation

The adjoint representation of the quantum group $U_q(\mathfrak{g})$ is the irreducible representation with the highest weight equal to the maximal root. Denote this representation by R. It has a basis $\{x_\mu, h_\alpha\}$ for $\mu \in \Phi, \alpha \in \Pi$ with the following action of the quantum group:

$$K_\alpha x_\mu = q^{(\alpha,\mu)} x_\mu, \qquad K_\alpha h_\beta = h_\beta; \tag{7}$$

E_α, F_α act by

$$\begin{array}{lll} E_\alpha x_\mu = 0, & F_\alpha x_\mu = 0 & \text{if } (\mu,\alpha) = 0; \\ E_\alpha x_\mu = 0, & F_\alpha x_\mu = x_{\mu-\alpha} & \text{if } (\mu,\alpha) = 1; \\ E_\alpha x_\mu = x_{\mu+\alpha}, & F_\alpha x_\mu = 0 & \text{if } (\mu,\alpha) = -1; \\ E_\alpha x_\alpha = 0, & F_\alpha x_\alpha = h_\alpha, & \\ E_\alpha x_{-\alpha} = h_\alpha, & F_\alpha x_{-\alpha} = 0; & \end{array} \tag{8}$$

and, for $\alpha, \beta \in \Pi$,

$$\begin{array}{lll} E_\alpha h_\beta & = & \begin{cases} (q+q^{-1})x_\alpha & \text{if } \alpha = \beta \\ x_\alpha & \text{if } (\beta,\alpha) = -1 \\ 0 & \text{otherwise} \end{cases} \\ F_\alpha h_\beta & = & \begin{cases} (q+q^{-1})x_{-\alpha} & \text{if } \alpha = \beta \\ x_{-\alpha} & \text{if } (\beta,\alpha) = -1 \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{9}$$

We will denote by R_μ the weight μ subspace of R. For $\mu \in \Phi$, the subspace R_μ is one-dimensional, while the dimension of R_0 is equal to the rank of $\mathfrak{g}$.

On R there is a τ-invariant semilinear form $<,>$,

$$< xa, b >=< a, \tau(x)b > \ \text{ for any } x \in U \text{ and } a, b \in R, \tag{10}$$

described in our basis by

$$\begin{array}{ll} < x_\mu, x_\mu >= 1, & \mu \in \Psi \\ < h_\alpha, h_\alpha >= 1 + q^2, & \alpha \in \Pi \\ < h_\alpha, h_\beta >= q & \text{if } (\alpha,\beta) = -1, \quad \alpha, \beta \in \Pi \\ < h_\alpha, h_\beta >= 0 & \text{if } (\alpha,\beta) = 0, \quad \alpha, \beta \in \Pi \end{array} \tag{11}$$

Distinct weight components of R are orthogonal with respect to this form. The basis $\{x_\mu, h_\alpha\}$ is called the canonical basis of R. It is a special case of the Lusztig-Kashiwara basis [L1],[L2],[Ka] (also see [J] for an introduction) in irreducible $U_q(\mathfrak{g})$ representations.

Let $l_\alpha \in R_0$ be defined by $< h_\beta, l_\alpha >= \delta_{\alpha,\beta}$. The basis $\{x_\mu, l_\alpha\}$ is dual to the canonical basis of R with respect to the semilinear form $<,>$. We call this basis the dual canonical basis of R.

Denote by I the $\mathbb{Z}[q, q^{-1}]$-submodule of R generated by elements of the dual canonical basis and by I' the $\mathbb{Z}[q, q^{-1}]$-submodule of R generated by canonical basis vectors. Note that I' is a $\mathbb{Z}[q, q^{-1}]$-submodule of I.

Let ψ_R be the $\mathbb{Q}$-linear involution $R \to R$ given by

$$\psi_R(x_\mu) = x_\mu, \quad \psi_R(h_\alpha) = h_\alpha, \quad \psi_R(fv) = \overline{f}\psi_R(v) \quad \text{for } f \in \mathbb{Q}(q), v \in U. \tag{12}$$

It is clear that $\psi_R(ax) = \psi(a)\psi_R(x)$ for $a \in U$ and $x \in R$ and that

$$< \psi_R x, \psi_R y > = < y, x > \quad \text{for} \quad x, y \in R. \tag{13}$$

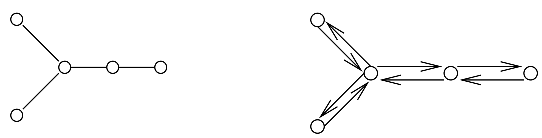
Let ω_R be the $\mathbb{Q}(q)$-linear involution $R \to R$ which takes x_μ to $x_{-\mu}$ and h_α to h_α. We have $\omega_R(ax) = \omega(a)\omega_R(x)$ for $a \in U$ and $x \in R$, and

$$< \omega_R x, \omega_R y > = < x, y > \quad \text{for} \quad x, y \in R. \tag{14}$$

Note that involutions ψ_R and ω_R preserve $I \subset R$.

3 Graphs and algebras

Let Γ be a connected graph without loops and multiple edges. Associated to Γ there is the double graph, $D\Gamma$, which has the same vertices as Γ and twice as many edges as Γ. Namely, each edge f of Γ is substituted by two oriented edges, which connect the same vertices as f and have opposite orientations. This construction is best illustrated by an example:



Graph Γ Double of Γ

Take the path algebra of $D\Gamma$. It is an algebra (over $\mathbb{C}$) spanned by all oriented paths in $D\Gamma$ with the multiplication given by concatenating paths. In particular, minimal idempotents correspond one-to-one to length 0 paths, i.e., to vertices of Γ. Since we assume that Γ has no multiple edges, we can describe a path by a list of vertices it travels through, thus, a path that starts at a point a, goes to b and then to c will be denoted $(a|b|c)$. If Γ has more than 2 vertices, denote by $A(\Gamma)$ the quotient algebra of this path algebra by the ideal generated by the following elements

(i) Paths $(a|b|c)$ for each triple of vertices a, b, c of Γ such that a, b are connected, b, c are connected and $a \neq c$,

(ii) Element $(a|b|a) - (a|c|a)$ whenever a is connected to both b and c.

If Γ consists of the single vertex only, define $A(\Gamma)$ as the algebra generated by 1 and X with $X^2 = 0$. If Γ consists of two points joined by a single edge, define $A(\Gamma)$ as the quotient of the path algebra of $D\Gamma$ by the two-sided ideal spanned by all paths of length greater than 2. We will call $A(\Gamma)$ the *zigzag algebra* of Γ.

If Γ has more than one vertex, $A(\Gamma)$ has a natural grading with paths of length k in degree k. If Γ have only one vertex, we introduce a grading on $A(\Gamma) = \mathbb{C}[X]/(X^2)$ by placing X in degree 2.

Denote by $v(\Gamma)$ the set of vertices of Γ and by $e(\Gamma)$ the set of its edges. For any Γ, the algebra $A(\Gamma)$ has nonzero components only in degrees $0, 1$ and 2, and the dimensions of these graded components are $v(\Gamma), 2e(\Gamma)$ and $v(\Gamma)$, respectively.

Proposition 1 $A(\Gamma)$ *is a graded symmetric algebra.*

Proof A finite-dimensional $\mathbb{C}$-algebra A is called symmetric if it possesses a nondegenerate symmetric trace map $A \to \mathbb{C}$. The trace map $\mathrm{tr} : A(\Gamma) \to \mathbb{C}$ is defined by sending each path of length 2 to 1 and all other paths to 0. Clearly, this map is symmetric, $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in A(\Gamma)$, and nondegenerate. $\square$

In particular, $A(\Gamma)$ is self-injective, i.e. $A(\Gamma)$ is injective as a left and right module over itself.

Let $\mathbb{C}$-Vect be the category of finite-dimensional graded complex vector spaces. The morphisms in this category are grading-preserving linear maps. If A is a finite-dimensional graded $\mathbb{C}$-algebra, denote by A-Mod

the abelian category of finite-dimensional graded A-modules and grading-preserving homomorphisms. If $M = \underset{n}{\oplus} M_n$ is a graded module over a graded algebra A, denote by $M\{k\}$ the module M with the grading shifted up by k, so that $M\{k\}_n = M_{n-k}$. Denote by $\{k\}$ the functor of shifting the grading up by k.

For a vertex $a \in v(\Gamma)$ denote by P_a the left projective module $A(\Gamma)e_a$ and by ${}_aP$ the right projective module $e_a A(\Gamma)$, where $e_a = (a)$ is the minimal idempotent equal to the zero length path which begins and ends in a. The left module P_a is spanned by all paths ending in a and ${}_aP$ is spanned by all paths starting at a. Any indecomposable graded projective left $A(\Gamma)$ module is isomorphic, up to a shift in the grading, to P_a for some vertex a. We have

$$ {}_aP \otimes_{A(\Gamma)} P_b \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}\{2\} & \text{if } \alpha = \beta \\ \mathbb{C}\{1\} & \text{if } (\beta, \alpha) = -1 \\ 0 & \text{otherwise} \end{cases} \tag{15} $$

For a vertex a of Γ consider functors

$$ \begin{array}{ll} T_a : A(\Gamma)\text{-Mod} \longrightarrow \mathbb{C}\text{-Vect}, & T_a(M) =_a P \otimes_{A(\Gamma)} M, \\ S_a : \mathbb{C}\text{-Vect} \longrightarrow A(\Gamma)\text{-Mod}, & S_a(V) = P_a \otimes_{\mathbb{C}} V. \end{array} \tag{16} $$

Lemma 1 *Functor T_a is right adjoint to S_a and left adjoint to $S_a\{-2\}$.*

This lemma follows from Proposition 1. Since the trace map of $A(\Gamma)$ has degree -2, this accounts for the appearance of the shift by $\{-2\}$ in the lemma. $\square$

4 Categorification

4.1 Grothendieck groups

If $\mathcal{B}$ is an abelian category, denote by $G(\mathcal{B})$ the Grothendieck group of $\mathcal{B}$. The Grothendieck group is the abelian group generated by symbols $[M]$ as M ranges over all objects of $\mathcal{B}$, with relations $[M_2] = [M_1] + [M_3]$ for each short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. An exact functor between abelian categories induces a homomorphism of their Grothendieck groups. Denote by $G'(\mathcal{B})$ the subgroup of $G(\mathcal{B})$ generated by symbols $[P]$ for all projective objects P of $\mathcal{B}$. We call $G'(\mathcal{B})$ the projective Grothendieck group.

If A is a graded algebra, the Grothendieck group and the projective Grothendieck group of the category A-Mod are $\mathbb{Z}[q, q^{-1}]$-modules, where multiplication by q corresponds to shift in the grading: $[M\{k\}] = q^k[M]$.

4.2 The category $\mathcal{C}$

We follow the notations from Section 2: $\mathfrak{g}$ is a simple simply-laced Lie algebra, Φ a root system of $\mathfrak{g}$ and Π a set of simple roots.

Let $\mathcal{C}_0$ be the category $A(\Gamma)$-Mod of finite dimensional graded left $A(\Gamma)$-modules, where Γ is the Dynkin diagram of $\mathfrak{g}$. For each root μ of the root system Φ, denote by $\mathcal{C}_\mu$ the category $\mathbb{C}$-Vect. For each μ, choose a one-dimensional vector space in $\mathcal{C}_\mu$, concentrated in degree 0, and denote it by $\mathbb{C}_\mu$.

Let the category $\mathcal{C}$ be the direct sum of $\mathcal{C}_0$ and categories $\mathcal{C}_\mu$ for all $\mu \in \Phi$:

$$ \mathcal{C} = \underset{\mu \in \Psi \cup \{0\}}{\oplus} \mathcal{C}_\mu \tag{17} $$

The vertices of the Dynkin diagram Γ are enumerated by the set of simple roots Π, thus for each simple root α there is an associated projective $A(\Gamma)$-module P_α and its simple quotient L_α.

There is an isomorphism between the Grothendieck group of $\mathcal{C}$ and the $\mathbb{Z}[q, q^{-1}]$-submodule I of R, given by sending $[\mathbb{C}_\mu] \in G(\mathcal{C}_\mu)$ to x_μ and $[L_\alpha]$ to ql_α. Thus, we identify images of simple objects of $\mathcal{C}$ in the Grothendieck group $G(\mathcal{C})$ with the elements of the dual canonical basis of R. Under this identification indecomposable projective modules are mapped to the (shifted by q) canonical basis vectors: $[P_\alpha] \longmapsto qh_\alpha$. We denote this isomorphism $G(\mathcal{C}) \cong I$ by ι and will use it to identify the Grothendieck group $G(\mathcal{C})$ with I, thus, we will write $[L_\alpha] = ql_\alpha, [P_\alpha] = qh_\alpha$, etc. This isomorphism restricts to an isomorphism between the projective Grothendieck group of $\mathcal{C}$ and the submodule I' of I.

The semilinear form $<,>$ on R can be interpreted as dimensions of homomorphism spaces. Namely, if $P \in \mathcal{C}$ is projective and $M \in \mathcal{C}$ is any module, we have

$$\sum_{i \in \mathbb{Z}} q^i \dim(\mathrm{Hom}_{\mathcal{C}}(P\{i\}, M)) =< [P], [M] > . \tag{18}$$

4.3 Functors

On the category $\mathcal{C}$ we define functors $\mathcal{E}_\alpha, \mathcal{F}_\alpha$ as follows. If $M \in \mathcal{C}_0$ then

$$\begin{aligned} &\mathcal{E}_\alpha(M) = ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes \mathbb{C}_\alpha \\ &\mathcal{F}_\alpha(M) = ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes \mathbb{C}_{-\alpha} \end{aligned} \tag{19}$$

Here and further the tensor products are over $\mathbb{C}$ unless indicated otherwise.

If $M \in \mathcal{C}_\mu$ and $\mu \neq 0$ then

$$\begin{array}{lll} \mathcal{E}_\alpha(M) = 0, & \mathcal{F}_\alpha(M) = 0 & \text{if } \ (\mu, \alpha) = 0 \\ \mathcal{E}_\alpha(M) = 0, & \mathcal{F}_\alpha(M) = M \otimes \mathbb{C}_{\mu-\alpha} & \text{if } \ (\mu, \alpha) = 1 \\ \mathcal{E}_\alpha(M) = M \otimes \mathbb{C}_{\mu+\alpha}, & \mathcal{F}_\alpha(M) = 0 & \text{if } \ (\mu, \alpha) = -1 \\ \mathcal{E}_\alpha(M) = 0, & \mathcal{F}_\alpha(M) = P_\alpha \otimes M\{-1\} & \text{if } \ \mu = \alpha \\ \mathcal{E}_\alpha(M) = P_\alpha \otimes M\{-1\}, & \mathcal{F}_\alpha(M) = 0 & \text{if } \ \mu = -\alpha \end{array} \tag{20}$$

Note that $\mathcal{E}_\alpha(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha}$ if $\mu + \alpha \in \Phi \cup \{0\}$ and $\mathcal{E}_\alpha(\mathcal{C}_\mu) = 0$ otherwise. Similarly, $\mathcal{F}_\alpha(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha}$ if $\mu - \alpha \in \Phi \cup \{0\}$ and $\mathcal{F}_\alpha(\mathcal{C}_\mu) = 0$ otherwise.

Introduce the functor $\mathcal{K}_\alpha : \mathcal{C} \to \mathcal{C}$ by

$$\mathcal{K}_\alpha(M) = M\{(\mu, \alpha)\} \quad \text{for} \quad M \in \mathcal{C}_\mu \tag{21}$$

Denote by $\mathcal{K}_\alpha^{-1}$ the inverse functor to $\mathcal{K}_\alpha$, thus, $\mathcal{K}_\alpha^{-1}(M) = M\{-(\mu, \alpha)\}$ for $M \in \mathcal{C}_\mu$.

Earlier we identified the Grothendieck group of $\mathcal{C}$ with the $\mathbb{Z}[q, q^{-1}]$-submodule I of R. The functors $\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{K}_\alpha$ are exact, commute with the shift functor $\{1\}$, and on the Grothendieck group of $\mathcal{C}$ act as the generators $E_\alpha, F_\alpha, K_\alpha$ of U.

4.4 Quantum group relations

Proposition 2 *There are functor isomorphisms*

$$\begin{aligned} &\mathcal{K}_\alpha \mathcal{K}_\alpha^{-1} \cong \mathrm{Id} \cong \mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha, \\ &\mathcal{K}_\alpha \mathcal{K}_\beta \cong \mathcal{K}_\beta \mathcal{K}_\alpha, \\ &\mathcal{K}_\alpha \mathcal{E}_\beta \cong \mathcal{E}_\beta \mathcal{K}_\alpha \{(\alpha, \beta)\}, \\ &\mathcal{K}_\alpha \mathcal{F}_\beta \cong \mathcal{F}_\beta \mathcal{K}_\alpha \{-(\alpha, \beta)\}, \\ &\mathcal{E}_\alpha \mathcal{F}_\beta \cong \mathcal{F}_\beta \mathcal{E}_\alpha \quad \textit{if} \quad \alpha \neq \beta, \\ &\mathcal{E}_\alpha \mathcal{E}_\beta \cong \mathcal{E}_\beta \mathcal{E}_\alpha \quad \textit{if} \quad (\alpha, \beta) = 0, \\ &\mathcal{F}_\alpha \mathcal{F}_\beta \cong \mathcal{F}_\beta \mathcal{F}_\alpha \quad \textit{if} \quad (\alpha, \beta) = 0, \\ &\mathcal{E}_\alpha^2 \mathcal{E}_\beta \oplus \mathcal{E}_\beta \mathcal{E}_\alpha^2 \cong (\mathrm{Id}\{1\} \oplus \mathrm{Id}\{-1\}) \mathcal{E}_\alpha \mathcal{E}_\beta \mathcal{E}_\alpha \quad \textit{if} \quad (\alpha, \beta) = -1 \\ &\mathcal{F}_\alpha^2 \mathcal{F}_\beta \oplus \mathcal{F}_\beta \mathcal{F}_\alpha^2 \cong (\mathrm{Id}\{1\} \oplus \mathrm{Id}\{-1\}) \mathcal{F}_\alpha \mathcal{F}_\beta \mathcal{F}_\alpha \quad \textit{if} \quad (\alpha, \beta) = -1 \end{aligned} \tag{22}$$

Proof: In a semisimple $\mathbb{C}$-linear category any linear relation in the endomorphism algebra of the Grothendieck group can be lifted into a functor isomorphism. Our category $\mathcal{C}$ has a huge semisimple direct summand, consisting of categories $\mathcal{C}_\mu$ for $\mu \neq 0$. Restricted to this subcategory, functor isomorphisms of the above proposition exist for obvious reasons. To prove functor isomorphisms (22) when the source category is $\mathcal{C}_\mu$ for some root μ and the target category is $\mathcal{C}_0$, it is enough to check, for each equation in (22), that the functors on the left and right hand sides of it, applied to the simple object $\mathbb{C}_\mu$, produce isomorphic objects. Since the target object is always projective, and isomorphism classes of projectives are determined by their images on the Grothendieck group, the claim follows. Functor isomorphisms (22) in the case when $\mathcal{C}_0$ is the source category and $\mathcal{C}_\mu$ the target category are proved similarly, by observing that each functor is isomorphic to the functor of tensoring with a graded right projective $A(\Gamma)$-module. $\square$

This takes care of all defining relations (1), save the following one:

$$E_\alpha F_\alpha - F_\alpha E_\alpha = \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}}. \tag{23}$$

The right hand side of (23) acts by $\frac{q^{(\mu,\alpha)} - q^{-(\mu,\alpha)}}{q-q^{-1}}$ on the weight subspace R_μ of R. If $i = (\mu, \alpha)$ is nonnegative, this quotient equals to $[i] = q^{i-1} + q^{i-3} + \cdots + q^{1-i}$, and is a Laurent polynomial in q with positive coefficients. Thus, on a weight subspace R_μ for $(\mu, \alpha) \geq 0$, we can rewrite (23) as

$$E_\alpha F_\alpha = F_\alpha E_\alpha + [(\mu, \alpha)] \tag{24}$$

Similarly, on the weight subspace R_μ for $(\mu, \alpha) \leq 0$ we can rewrite the equation (23) as

$$E_\alpha F_\alpha + [-(\mu, \alpha)] = F_\alpha E_\alpha. \tag{25}$$

Both left and right hand sides of the two equations above have only positive coefficients, and it is in this form that the equation (23) lifts into a functor isomorphism. To state the isomorphism, we will denote by $\mathrm{Id}^{[j]}$ for $j \geq 0$ the functor $\mathrm{Id}\{j-1\} \oplus \mathrm{Id}\{j-3\} \oplus \cdots \oplus \mathrm{Id}\{1-j\}$ in the category $\mathcal{C}_\mu$.

Proposition 3 *For $\mu \in \Phi \cup \{0\}$ there is an isomorphism of functors in the category $\mathcal{C}_\mu$*

$$\mathcal{E}_\alpha \mathcal{F}_\alpha \cong \mathcal{F}_\alpha \mathcal{E}_\alpha \oplus \mathrm{Id}^{[(\mu,\alpha)]} \quad \textit{if} \quad (\mu, \alpha) \geq 0, \tag{26}$$

$$\mathcal{E}_\alpha \mathcal{F}_\alpha \oplus \mathrm{Id}^{[-(\mu,\alpha)]} \cong \mathcal{F}_\alpha \mathcal{E}_\alpha \quad \textit{if} \quad (\mu, \alpha) \leq 0. \tag{27}$$

Proof is left to the reader. □

In the next two subsections we show that the three automorphisms and antiautomorphisms of U, defined in Section 2.1, can be interpreted as various dualities in the category $\mathcal{C}$. We then explain how the braid group action on the weight 0 subspace R_0 of R lifts to a braid group action on the derived category of $A(\Gamma)$-modules. This plentitude of interesting structures in the category $\mathcal{C}$ clearly points to the naturality and uniqueness of $\mathcal{C}$. Any other realization of the adjoint representation R as the Grothendieck group of an abelian category will fail to be as rich as the one that we describe here.

4.5 Adjointness and dualities

Adjointness and the antiautomorphism τ. Functors $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ have left and right adjoints:

Proposition 4 *The functor $\mathcal{E}_\alpha$ is left adjoint to $\mathcal{F}_\alpha \mathcal{K}_\alpha^{-1}\{1\}$, the functor $\mathcal{F}_\alpha$ is left adjoint to $\mathcal{E}_\alpha \mathcal{K}_\alpha\{1\}$ and $\mathcal{K}_\alpha$ is left adjoint to $\mathcal{K}_\alpha^{-1}$.*

Proof This proposition easily reduces to Lemma 1. □

Comparing this proposition with the formula (2) for the antiautomorphism τ, we see that τ *becomes the operation of taking the right adjoint functor.* Let V^{ad} denote the right adjoint of a functor V, when the right adjoint exists. Suppose that functors V_1 and V_2 are composable and admit right adjoints. Then $(V_1 V_2)^{\mathrm{ad}} \cong V_2^{\mathrm{ad}} V_1^{\mathrm{ad}}$, i.e. passing to adjoints interchanges the order in the product of functors. Correspondingly, τ is an antiautomorphism, $\tau(ab) = \tau(b)\tau(a)$.

Formula (10) now has a nice categorical interpretation. Namely, earlier we found that the semilinear form $<,>$ computes the dimension of the spaces of morphisms from a projective to an arbitrary module in $\mathcal{C}$ (see formula (18)). If V denotes an arbitrary product of functors $\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{K}_\alpha^{\pm 1}$ and $\{\pm 1\}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(V(P), M) \cong \mathrm{Hom}_{\mathcal{C}}(P, V^{\mathrm{ad}}(M)) \tag{28}$$

for any objects P, M of $\mathcal{C}$. For what follows, take P a projective module. Then $V(P)$ is also projective, so that, after an appropriate summation over all shifts by $\{i\}$, as in (18), we derive the formula (10) for the τ-invariance of the semilinear form $<,>$.

ψ **and the contravariant duality functor.** Let $* : \mathbb{C}\text{-Vect} \to \mathbb{C}\text{-Vect}$ be the contravariant functor in the category of graded vector spaces which takes a vector space V to its dual $V^* = \mathrm{Hom}(V, \mathbb{C})$. Note that $(\mathbb{C}\{i\})^* \cong \mathbb{C}\{-i\}$.

Let χ be the antiinvolution of $A(\Gamma)$ which takes a path $(a_1|a_2|\dots|a_k)$ in $A(\Gamma)$ to the opposite path $(a_k|\dots|a_2|a_1)$.

Let $\Psi : \mathcal{C} \to \mathcal{C}$ be the following contravariant duality functor. For $M \in \mathrm{Ob}(\mathcal{C}_\mu)$ where $\mu \neq 0$ we set $\Psi M = M^* \in \mathcal{C}_\mu$. For $M \in \mathcal{C}_0 = A(\Gamma)$-Mod the graded vector space M^* has the structure of a right graded $A(\Gamma)$-module. We use the antiinvolution χ to make it into a left graded $A(\Gamma)$-module. Thus, $\Psi M = M^* \in \mathcal{C}_0$.

The following proposition is obvious.

Proposition 5 *1. On the Grothendieck group of $\mathcal{C}$ functor Ψ acts as the involution ψ_R of R, defined by the formula (12).*

2. There are functor isomorphisms

$$\Psi\mathcal{E}_\alpha \cong \mathcal{E}_\alpha\Psi, \ \Psi\mathcal{F}_\alpha \cong \mathcal{F}_\alpha\Psi, \ \Psi K_\alpha \cong K_\alpha^{-1}\Psi, \ \Psi\{i\} \cong \{-i\}\Psi. \tag{29}$$

3. Ψ is an involution, i.e. Ψ^2 is isomorphic to the identity functor.

Note that functor isomorphisms (29) lift the defining relations (3) of ψ. Thus, Ψ lifts the involution ψ of U as well as the involution ψ_R of R. Since Ψ is a contravariant equivalence of $\mathcal{C}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\Psi M, \Psi N) \cong \mathrm{Hom}_{\mathcal{C}}(N, M), \qquad \text{for} \qquad M, N \in \mathrm{Ob}(\mathcal{C}), \tag{30}$$

which in the Grothendieck group of $\mathcal{C}$ translates into the formula (13).

ω_R **as an automorphism of** $\mathcal{C}$. Let Ω be the following self-equivalence of the category $\mathcal{C}$:

(i) Ω, restricted to $\mathcal{C}_0$, is the identity functor,

(ii) Ω, restricted to $\mathcal{C}_\mu$ for $\mu \in \Phi$, is an equivalence of categories $\mathcal{C}_\mu \to \mathcal{C}_{-\mu}$, coming from the identification of both $\mathcal{C}_\mu$ and $\mathcal{C}_{-\mu}$ with the category $\mathbb{C}$-Vect.

The following proposition is obvious.

Proposition 6 *1. On the Grothendieck group of $\mathcal{C}$ functor Ω acts as the involution ω_R of R (ω_R was defined in Section 2.2).*

2. There are functor isomorphisms

$$\Omega\mathcal{E}_\alpha \cong \mathcal{F}_\alpha\Omega, \quad \Omega\mathcal{F}_\alpha \cong \mathcal{E}_\alpha\Omega, \quad \Omega\mathcal{K}_\alpha \cong \mathcal{K}_\alpha^{-1}\Omega. \tag{31}$$

3. Ω is an involution, i.e. Ω^2 is isomorphic to the identity functor.

Functor Ω corresponds to the involution ω of U and ω_R of R. In particular, formula (4) becomes functor isomorphisms (31). Since Ω is an equivalence, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\Omega M, \Omega N) \cong \mathrm{Hom}_{\mathcal{C}}(M, N), \qquad M, N \in \mathrm{Ob}(\mathcal{C}), \tag{32}$$

which in the Grothendieck group of $\mathcal{C}$ translates into the formula (14).

4.6 The braid group action

For a graph Γ denote by $\mathrm{Br}(\Gamma)$ the braid group associated to Γ. It has generators σ_a for each vertex a of Γ and relations

$$\begin{aligned} \sigma_a\sigma_b\sigma_a &= \sigma_b\sigma_a\sigma_b && \text{if } a \text{ and } b \text{ are joined by an edge,} \\ \sigma_a\sigma_b &= \sigma_b\sigma_a && \text{otherwise.} \end{aligned}$$

Every finite dimensional representation V of $U_q(\mathfrak{g})$ comes equipped with a natural action of the braid group $\mathrm{Br}(\Gamma)$, where Γ is the Dynkin diagram of $\mathfrak{g}$ (see Jantzen [J], for instance). This action preserves the

weight 0 subspace of V. When $V = R$ is the adjoint representation, the braid group action on the weight 0 subspace R_0 has a particularly simple form, with σ_α acting as $qE_\alpha F_\alpha - 1$. This action can be categorified. Indeed, the functor counterpart of the operator $qE_\alpha F_\alpha$ is $\mathcal{E}_\alpha \mathcal{F}_\alpha\{1\}$. Restricted to the subcategory $\mathcal{C}_0$ of $\mathcal{C}$ this functor is isomorphic to the functor of tensoring with the $A(\Gamma)$-bimodule $P_\alpha \otimes {}_\alpha P$. There is a canonical bimodule map

$$\zeta \quad : \quad P_\alpha \otimes {}_\alpha P \longrightarrow A(\Gamma), \qquad \zeta(l_1 \otimes l_2) = l_1 l_2 \tag{33}$$

where $l_1 \in P_\alpha$ is a path with target α and $l_2 \in {}_\alpha P$ is a path with source α.

Form the bounded derived category $D^b(A(\Gamma)\text{-Mod})$ of the abelian category $A(\Gamma)$-Mod. Let

$$\Sigma_a \quad : \quad D^b(A(\Gamma)\text{-Mod}) \longrightarrow D^b(A(\Gamma)\text{-Mod}) \tag{34}$$

be the functor of tensoring with the complex

$$0 \longrightarrow P_a \otimes {}_a P \xrightarrow{\zeta} A(\Gamma) \longrightarrow 0 \tag{35}$$

of $A(\Gamma)$-bimodules, where a is a vertex of Γ.

Proposition 7 *Functors Σ_a are invertible and there are functor isomorphisms*

$$\begin{aligned} \Sigma_a \Sigma_b \Sigma_a &\cong \Sigma_b \Sigma_a \Sigma_b && \textit{if } a \textit{ and } b \textit{ are joined by an edge,} \\ \Sigma_a \Sigma_b &\cong \Sigma_b \Sigma_a && \textit{otherwise.} \end{aligned}$$

Proofs in [KS] and [ST] for the case when Γ is the Dynkin diagram of $\mathfrak{sl}_n$ generalize to arbitrary graphs without difficulty. When the graph is a chain, this braid group action was also considered by R.Rouquier and A.Zimmermann [RZ].

□

Thus, the braid group $\mathrm{Br}(\Gamma)$ acts on the derived category of the category of graded $A(\Gamma)$-modules. When Γ is a finite Dynkin graph, we obtain a braid group action on the derived category $D^b(\mathcal{C}_0)$ which lifts the braid group action on the weight 0 subspace of the adjoint representation. It is proved in [KS] that this action is faithful if Γ is the Dynkin diagram of $\mathfrak{sl}_n$.

Braid group and invertibility relations come from homotopy equivalences between complexes of bimodules which are tensor products of complexes (35) and of similar complexes describing the inverse functors. In particular, functors of tensoring with (35) also define a braid group action in various homotopy categories of complexes of $A(\Gamma)$-modules.

Note that the braid group action on the W-invariant subspace $\oplus_{\mu \in \Psi} R_\mu$ of R can be trivially lifted to the action on the derived category $D^b(\oplus_{\mu \in \Psi} \mathcal{C}_\mu)$, the latter action given by permutations of categories $\mathcal{C}_\mu$ and shifts in the derived category. Thus, the braid group acts in the derived category $D^b(\mathcal{C})$. On the Grothendieck group this action descends to the standard action of the braid group in the adjoint representation R of $U_q(\mathfrak{g})$.

Our braid group actions generalize to deformations of zigzag algebras with type (ii) (see Section 3) defining relations $(a|b|a) = \nu^a_{b,c}(a|c|a)$, where $\nu^a_{b,c} \in \mathbb{C}^*$ and satisfy compatibility relations $\nu^a_{b,c}\nu^a_{c,b} = 1$ and $\nu^a_{b,c}\nu^a_{c,d}\nu^a_{d,b} = 1$ whenever a is connected to b, c, and d. In a zigzag algebra $\nu^a_{b,c} = 1$ for all possible triples (a, b, c).

We call these deformations *skew-zigzag algebras* and denote by $A_\nu(\Gamma)$. They are Frobenius but not, in general, symmetric algebras. Rescaling $(a|b)$'s changes coefficients $\nu^a_{b,c}$, and the moduli space of skew-zigzag algebras is naturally isomorphic to $H^1(\Gamma, \mathbb{C}^*)$. In particular, if Γ is a tree, all of its skew-zigzag algebras are isomorphic.

Derived and homotopy categories of modules over skew-zigzag algebras admit braid group actions that are constructed in the same way as for zigzag algebras.

5 Zigzag algebras and their representations

Let B be a finite dimensional algebra over $\mathbb{C}$. There is an obvious B-bimodule structure on $B^* = \mathrm{Hom}_{\mathbb{C}}(B, \mathbb{C})$. Define the algebra $T(B) = B \oplus B^*$ with the multiplication $(x, f)(y, g) = (xy, fy + xg)$ for $x, y \in B$ and

$f, g \in B^*$. Then $T(B)$ is an associative algebra with a nondegenerate symmetric trace map $\mathrm{tr} : T(B) \to \mathbb{C}$. It is called *the trivial extension algebra of* B.

Let Γ be a graph as before and denote by Γ^0 any of the oriented graphs obtained by picking an orientation of each edge of Γ. Let $B(\Gamma^0)$ be the path algebra of Γ^0. A theorem of Gabriel says that $B(\Gamma^0)$ has finite representation type if and only if Γ is a finite Dynkin diagram, in which case there is a natural one-to-one correspondence between indecomposable representations of $B(\Gamma^0)$ and positive roots of the root system associated to Γ.

Let $B^{\mathrm{red}}(\Gamma^0)$ be the path algebra of Γ^0, quotiented out by all paths of length greater than 1. Let Γ^1 be the graph Γ with the orientation opposite to that of Γ^0. Algebras $B(\Gamma^0)$ and $B^{\mathrm{red}}(\Gamma^0)$ are graded by the length of paths and we have the following obvious (here and further we refer the reader to [BGS] for definition and properties of Koszul algebras)

Proposition 8 *Algebras* $B^{\mathrm{red}}(\Gamma^0)$ *and* $B(\Gamma^1)$ *are Koszul dual.* $\square$

There is a natural inclusion of algebras $B^{\mathrm{red}}(\Gamma^0) \hookrightarrow A(\Gamma)$. Indeed, minimal idempotents of $B^{\mathrm{red}}(\Gamma^0)$ and of $A(\Gamma)$ are identified with vertices of Γ and every oriented edge of Γ^0 is also an edge of the double $D\Gamma$ of Γ. This correspondence extends to the abovementioned inclusion $B^{\mathrm{red}}(\Gamma^0) \hookrightarrow A(\Gamma)$. Note that as a vector space $A(\Gamma)$ decomposes into the direct sum of $B^{\mathrm{red}}(\Gamma^0)$ and the subspace spanned by edges of $D\Gamma$ which are complementary to the ones of Γ^0 and by length 2 paths $(a|b|a)$, one for each vertex a of Γ. This complementary subspace, considered as a $B^{\mathrm{red}}(\Gamma^0)$-bimodule, is canonically isomorphic to $(B^{\mathrm{red}}(\Gamma^0))^*$, and, therefore, we derive

Proposition 9 *For any orientation* Γ^0 *of the graph* Γ, *the algebra* $A(\Gamma)$ *is isomorphic to the trivial extension algebra of* $B^{\mathrm{red}}(\Gamma^0)$.

Assume now that Γ is a tree. Choose an orientation Γ^0 of Γ such that each vertex of Γ is either a source or a sink. Equivalently, Γ^0 has no oriented paths of length 2. The graph Γ has exactly two such orientations. Notice that $B^{\mathrm{red}}(\Gamma^0)$ is isomorphic to the path algebra of Γ^0, since the later does not contain any paths of length greater than 1.

Below, when we talk about correspondences between indecomposable representations, we actually mean correspondences between isomorphism classes of indecomposable representations. For brevity, "isomorphism classes" will be omitted everywhere.

Proposition 10 *There is a natural two-to-one correspondence between indecomposable representations of* $A(\Gamma)$ *and indecomposable representations of* $B(\Gamma^0)$.

Note: The referee pointed out that this is a known result, proved in Tachikawa [Ta]. We retained the proof for completeness.

Proof: Let M be a representation of $A(\Gamma)$. If for some vertex a of Γ the module M is not annihilated by $(a|b|a)$, then it is easy to see that M contains a projective module as a direct summand. Therefore, we assume from now on that M is annihilated by $(a|b|a)$ for each vertex a. The module M decomposes as a direct sum of vector spaces $(a)M$, over all a. Denote by v_+ the set of vertices of Γ which are source vertices of Γ^0 and by v_- the set of sinks of Γ^0. The disjoint union of v_+ and v_- is the set of vertices of Γ.

For each edge (a, b) of Γ there are maps $(a)M \to (b)M$ and $(b)M \to (a)M$, given by left multiplication by $(b|a)$ and $(a|b)$, respectively. For each a denote by $Ker(a)$ the subspace of $(a)M$ which is the intersection of kernels of all maps $(a)M \to (b)M$, as b varies over all vertices adjoint to a. For each a choose an arbitrary complement $Comp(a)$ to $Ker(a)$ in $(a)M$. Let

$$\begin{aligned} M_0 &= \underset{a\in v_+}{\oplus} Comp(a) \oplus (\underset{a\in v_-}{\oplus} Ker(a)) \\ M_1 &= \underset{a\in v_-}{\oplus} Comp(a) \oplus (\underset{a\in v_+}{\oplus} Ker(a)) \end{aligned}$$

Since M is annihilated by paths $(a|b|a)$, for all pairs (a, b), vector subspaces M_0 and M_1 are actually submodules of M. Submodule M_0 has the following structure: we start with a $B(\Gamma^0)$ module and extend the action to the whole $A(\Gamma)$ by declaring that the standard complement $B(\Gamma^0)^*$ to $B(\Gamma^0)$ in $A(\Gamma)$ acts by 0.

In particular, M_0 is indecomposable if and only if it is indecomposable as a $B(\Gamma^0)$-module. Similarly, M_1 is indecomposable if and only if it is indecomposable as a $B(\Gamma^1)$-module.

We thus get a map from the disjoint union of indecomposable representations of $B(\Gamma^0)$ and $B(\Gamma^1)$ to the set of indecomposable representations of $A(\Gamma)$. This map restricts to a bijection from the set of non-irreducible indecomposable representations of $B(\Gamma^0)$ and $B(\Gamma^1)$ to the set of indecomposable $A(\Gamma)$ representations which are neither irreducible nor projective. On irreducible representations this map is two-to-one. Since there is a one-to-one correspondence between irreducible and projective representations of $A(\Gamma)$, we can modify this map to be a bijection between the disjoint union of indecomposable representations of $B(\Gamma^0)$ and $B(\Gamma^1)$ to the set of indecomposable representations of $A(\Gamma)$. There is a canonical modification which preserves the symmetry between Γ^0 and Γ^1. Namely, if $a \in v_+$, send the simple $B(\Gamma^0)$ module $\mathbb{C}(a)$ to the projective $A(\Gamma)$-module P_a and the simple $B(\Gamma^1)$-module $\mathbb{C}(a)$ to the simple quotient of P_a. For $a \in v_-$, send the simple $B(\Gamma^1)$ module $\mathbb{C}(a)$ to the projective $A(\Gamma)$-module P_a and the simple $B(\Gamma^0)$-module $\mathbb{C}(a)$ to the simple quotient of P_a.

To finish the proof, note that there is a bijection between indecomposable representations of $B(\Gamma^0)$ and $B(\Gamma^1)$ given by passing to the dual vector space. $\square$

For a finite Dynkin graph Γ indecomposable representations of $B(\Gamma^0)$ are in a one-to-one correspondence with positive roots of the root system associated to Γ. We get a

Corollary 1 *If Γ is a finite Dynkin graph then indecomposable representations of $A(\Gamma)$ are in a one-to-one correspondence with roots of Γ.*

6 Zigzag algebras for affine Dynkin diagrams and the McKay correspondence

6.1 Zigzag algebras, Koszulity and quantum Cartan matrices

If Γ has more than one vertex, $A(\Gamma)$ is generated by elements of degree 0 and 1. If Γ has more than two vertices, $A(\Gamma)$ is a quadratic algebra. Roberto Martínez-Villa [MV] found a surprising characterization of Dynkin diagrams in terms of zigzag algebras $A(\Gamma)$:

Proposition 11 *Algebra $A(\Gamma)$ is Koszul if and only if Γ is not a finite Dynkin graph.*

The quadratic dual $A^!(\Gamma)$ of $A(\Gamma)$ is isomorphic to the quotient algebra of the path algebra of the double graph $D\Gamma$ by relations $\sum_b (a|b|a) = 0$ where we sum over all vertices b of Γ adjacent to a. If Γ is bipartite, $A^!(\Gamma)$ is isomorphic to the preprojective algebra of the source-sink oriented graph Γ (see Reiten [R] for an introduction to preprojective algebras).

Note that the algebra $A^!(\Gamma)$ is finite-dimensional if and only if Γ is a finite Dynkin diagram, while $A(\Gamma)$ is finite-dimensional for any graph Γ.

The Cartan matrix of a finite-dimensional $\mathbb{C}$-algebra B has rows and columns enumerated by isomorphism classes of indecomposable projective left B-modules, and its entries are dimensions of homomorphism spaces between these projectives. Algebra $A(\Gamma)$ is graded, and its Cartan matrix has coefficients which are polynomials in q,

$$c_{ab} = \sum_{i \geq 0} q^i \dim(\mathrm{Hom}_{A(\Gamma)\text{-Mod}}(P_a\{i\}, P_b)), \tag{36}$$

where a and b are vertices of Γ. Clearly, $c_{a,a} = 1 + q^2$, $c_{a,b} = q$ if a and b are connected by an edge and $c_{a,b} = 0$ otherwise.

On the other hand, in the theory of Lie algebras the expression "Cartan matrix" is used to denote a matrix naturally associated to a graph Γ. This matrix has 2 as each diagonal entry, -1 on the intersection of the column a and row b if a and b are joined by an edge and 0 otherwise. Now, if we set $q = -1$, the Cartan matrix of the algebra $A(\Gamma)$ is equal to the Cartan matrix of the graph Γ.

The determinant of the Cartan matrix of Γ is 0 if Γ is an affine Dynkin diagram. However, the determinant of the Cartan matrix of $A(\Gamma)$ is a nonzero polynomial in q for any graph Γ. We will call the Cartan matrix

of $A(\Gamma)$ *the quantum Cartan matrix* of Γ and denote it by $C(\Gamma)$. This matrix is always invertible in the field $\mathbb{Q}(q)$ of rational functions in q.

Let $P^!_a = A^!(\Gamma)(a)$ be the indecomposable projective left $A^!(\Gamma)$ module associated to the minimal idempotent (a) of $A^!(\Gamma)$, for a vertex a of Γ. Let $C^!(\Gamma)$ be the Cartan matrix of $A^!(\Gamma)$. Its (a,b)-entry is

$$c^!_{a,b} = \sum_{i\geq 0} q^i \dim(\mathrm{Hom}(P^!_a\{i\}, P^!_b)) \tag{37}$$

As we have already mentioned, if Γ is not a finite Dynkin diagram, the zigzag algebra $A(\Gamma)$ is Koszul. This and the acyclicity of the Koszul complex implies

Proposition 12 *If Γ is not a finite Dynkin diagram, the matrix $C^!(\Gamma)$, with q changed to $-q$ everywhere, is the inverse matrix of $C(\Gamma)$:*

$$C_q(\Gamma)C^!_{-q}(\Gamma) = \mathrm{I}.$$

Corollary 2 *If Γ if not a finite Dynkin diagram, the entries of the inverse quantum Cartan matrix $C(\Gamma)^{-1}$ are power series in $-q$ with nonnegative coefficients.*

Quantum Cartan matrices for affine Dynkin diagrams Γ appear in the paper of Lusztig and Tits [LT] which contains a simple algorithm for computing the inverse of a Cartan matrix. Since the Cartan matrix of an affine Dynkin diagram is not invertible, Lusztig and Tits change all diagonal entries from 2 to $T+T^{-1}$. If we set $T=-q$ and multiply each entry by $-q$, we get our quantum Cartan matrix $C(\Gamma)$.

6.2 A digression: Koszul duality for cross-products

Let V be a finite-dimensional complex vector space and G a finite group acting on V. Let $SV = \underset{i\geq 0}{\oplus} S^iV$ be the polynomial algebra of V and $S(V,G) = SV \otimes \mathbb{C}[G]$ be the cross-product algebra. The multiplication in $S(V,G)$ is given by

$$(a\otimes g)(b\otimes h) = ag(b)\otimes gh, \quad \text{for} \quad a,b\in SV \text{ and } g,h\in G. \tag{38}$$

$S(V,G)$ is a $\mathbb{Z}$-graded algebra, $S(V,G) = \underset{i\geq 0}{\oplus} S^i(V,G)$ where $S^i(V,G) = S^iV\otimes\mathbb{C}[G]$. Note that $S^0(V,G) = \mathbb{C}[G]$ is semisimple and that SV and $\mathbb{C}[G]$ are subalgebras of $S(V,G)$.

The left $S(V,G)$-module $\mathbb{C}[G] = S(V,G)/S^{>0}(V,G)$ has a projective resolution by left $S(V,G)$-modules

$$\ldots \longrightarrow SV\otimes\Lambda^2V\otimes\mathbb{C}[G] \longrightarrow SV\otimes V\otimes\mathbb{C}[G] \longrightarrow \mathbb{C}[G] \longrightarrow 0 \tag{39}$$

where the differential $\partial : SV\otimes\Lambda^iV\otimes\mathbb{C}[G] \longrightarrow SV\otimes\Lambda^{i-1}V\otimes\mathbb{C}[G]$ is

$$\partial(x\otimes y_1\ldots y_i\otimes g) = \sum_{j=1}^{i}(-1)^j xy_j\otimes y_1\ldots y_{j-1}y_{j+1}\ldots y_i\otimes g \tag{40}$$

and the left $S(V,G)$-module structure of $SV\otimes\Lambda^iV\otimes\mathbb{C}[G]$ is given by

$$(x_1\otimes g)(x_2\otimes y\otimes h) = x_1g(x_2)\otimes g(y)\otimes gh, \quad x_1,x_2\in SV,\ y\in\Lambda^iV,\ g,h\in G. \tag{41}$$

It is easy to see that $SV\otimes\Lambda^iV\otimes\mathbb{C}[G]$ is a free $S(V,G)$-module of rank equal to the dimension of Λ^iV. In particular, (39) is a graded projective resolution and its i-th term is generated by the subspace $1\otimes\Lambda^iV\otimes 1$ of elements of degree i. Therefore, we obtain

Proposition 13 *Graded algebra $S(V,G)$ is quadratic and Koszul.*

Define $\Lambda(V,G)$ as the cross-product algebra $\Lambda V\otimes\mathbb{C}[G]$ where ΛV is the exterior algebra of V, and the product in $\Lambda(V,G)$ is also given by the equation (38). Algebra $\Lambda(V,G)$ has a natural grading coming from the grading of the exterior algebra ΛV.

Consider the algebra $\Lambda(V^*,G)$ where V^* is the dual representation of G. The differential (40) commutes with the natural right action of $\Lambda(V^*,G)$ on $SV\otimes\Lambda V\otimes\mathbb{C}[G]$ (the convolution action of $\Lambda(V^*)$ on ΛV together with the right multiplication in the group algebra), and we can view (39) as the Koszul complex for the pair of Koszul dual algebras $S(V,G)$ and $\Lambda(V^*,G)$.

Corollary 3 *The graded algebra $\Lambda(V^*, G)$ is quadratic and Koszul. Its Koszul dual is isomorphic to $S(V, G)$.*

Remark Strictly speaking, since the group ring of G is in general not commutative, we should distinguish between left and right quadratic duals (see [BGS], Section 2.8). However, in our case, left and right quadratic duals are isomorphic, and that distinction is not necessary.

Remark Suitably formulated, the duality of Corollary 3 holds for any reductive group G and a finite-dimensional representation V of G.

Proposition 14 *Algebra $\Lambda(V, G)$ is Frobenius. If $\dim V$ is odd and $G \subset SL(V)$ then $\Lambda(V, G)$ is symmetric. If $\dim V$ is even, $G \subset SL(V)$ and G contains a central element h acting as $-\mathrm{Id}$ on V then $\Lambda(V, G)$ is symmetric.*

Proof: Let $n = \dim V$ and $x \in \Lambda^n V, x \neq 0$. The trace map tr which is 0 on $\Lambda^i V \otimes \mathbb{C}[G]$ for $i < n$ and $\mathrm{tr}(x \otimes g) = \delta_{g,1}$ makes $\Lambda(V, G)$ Frobenius. This trace is symmetric when (V, G) satisfies the first condition of the proposition. If (V, G) satisfies the second condition, set instead $\mathrm{tr}(x \otimes g) = \delta_{g,h}$, for $g \in G$. $\square$

6.3 Zigzag algebras and resolutions of simple singularities

Let G be a finite subgroup of $SU(2)$ and $X = \widetilde{\mathbb{C}^2/G}$ the minimal resolution of the quotient $\mathbb{C}^2/G$. The singular fiber is a union of projective lines, to which we associate a graph with one vertex for each projective line and an edge for each pair of intersecting projective lines. We denote this graph by $\Gamma(G)$, or simply by Γ. This graph is a finite Dynkin diagram, and the construction above gives a well-known bijection between finite subgroups of $SU(2)$ and simple simply-laced Lie algebras.

To catch a glimpse of zigzag algebras, form the direct sum $\mathcal{O}'$ of the structure sheaves of projective lines in X.

Proposition 15 *Graded algebras $\mathrm{Ext}^*_{\mathrm{Coh}(X)}(\mathcal{O}', \mathcal{O}')$ and $A(\Gamma)$ are naturally isomorphic.*

The Ext groups are computed in the category of coherent sheaves on X. This proposition also appears in [ST]. $\square$

We will say that G is *binary* if G has even order, equivalently, if G contains $-I$, the only order 2 element of $SU(2)$. A non-binary subgroup is necessarily an odd order cyclic group.

The action of G on $\mathbb{C}^2$ naturally extends to an action of G on the exterior algebra on two generators. Form the cross-product algebra $\Lambda(\mathbb{C}^2, G)$. This is a finite-dimensional algebra, Morita equivalent to an algebra described by a finite quiver with relations. Vertices of this quiver are in a bijection with irreducible representations of G, since the zero degree component of $\Lambda(\mathbb{C}^2, G)$ is isomorphic to the group algebra of G. Denote by V_a the irreducible representation of G associated with the vertex a. Recall that, as observed by McKay [McK], if to a $G \subset SU(2)$ we associate a graph with vertices – irreducible representations of G and with the number of edges connecting a and b equal to the multiplicity of V_b in the tensor product $\mathbb{C}^2 \otimes V_a$, we obtain an affine Dynkin diagram. We denote this diagram by Γ^{aff}.

In our case, the oriented graph underlying the quiver algebra of $\Lambda(\mathbb{C}^2, G)$ is the oriented double of Γ^{aff}. Since

(a) $\Lambda(\mathbb{C}^2, G)$ has the top degree component in degree two,

(b) if G is binary, $\Lambda(\mathbb{C}^2, G)$ has a symmetric nondegenerate graded trace (Proposition 14),

we easily deduce

Proposition 16 *If G is binary, $\Lambda(\mathbb{C}^2, G)$ is Morita equivalent to the zigzag algebra of the affine Dynkin diagram Γ^{aff}.*

G is non-binary iff it is cyclic of odd order. In general, if G is cyclic of order n then $\Lambda(\mathbb{C}^2, G)$ is isomorphic to the quiver algebra of the quiver with vertices $1, 2, \ldots, n$, edges $(i|i \pm 1)$ modulo n, and relations

$$(i|i+1|i+2) = (i|i-1|i-2) = 0, \qquad (i|i-1|i) + (i|i+1|i) = 0. \tag{42}$$

This is an example of a skew-zigzag algebra (see Section 4.6). It is isomorphic to a zigzag algebra if n is even.

The Koszul dual statement to Proposition 16 can be formulated as

Proposition 17 *If G is binary, $S(\mathbb{C}^2, G)$ is Morita equivalent to $A^!(\Gamma^{\mathrm{aff}})$, the latter isomorphic to the preprojective algebra of* Γ^{aff}.

Kapranov and Vasserot [KV] proved that the derived category of coherent sheaves on the minimal resolution X is equivalent to the derived category of finitely-generated $S(\mathbb{C}^2, G)$-modules. Notice that X comes with a canonical action of $\mathbb{C}^*$. We conjecture that the derived category of $\mathbb{C}^*$-equivariant sheaves on X is equivalent to the derived category of graded finitely-generated $S(\mathbb{C}^2, G)$-modules. If true, then, in view of the Koszul duality between $S(\mathbb{C}^2, G)$ and $\Lambda(\mathbb{C}^2, G)$, we would get an equivalence of categories between the derived category of $\mathbb{C}^*$-equivariant sheaves on X and the derived category of graded $A(\Gamma^{\mathrm{aff}})$ modules (Proposition 15 picks up part of this equivalence).

Proposition 18 *The affine braid group associated to the affine Dynkin diagram Γ^{aff} acts in the derived category of $\Lambda(\mathbb{C}^2, G)$-modules, in the derived category of $S(\mathbb{C}^2, G)$-modules, and in the derived category of coherent sheaves on the minimal resolution of $\mathbb{C}^2/G$.*

Proof: For $\Lambda(\mathbb{C}^2, G)$ and binary G this follows from propositions 16 and 7. For non-binary G the algebra $\Lambda(\mathbb{C}^2, G)$ is skew-zigzag and there is a braid group action in the derived category according to the remarks at the end of Section 4.6. For $S(\mathbb{C}^2, G)$ we get the braid group action by Koszul duality. For a vertex $a \in \Gamma^{\mathrm{aff}}$ the braid σ_a acts by taking $M \in D^b(S(\mathbb{C}^2, G)\text{-mod})$ to the cone of the evaluation map of complexes

$$V'_a \otimes \mathrm{RHom}(V'_a, M) \longrightarrow M,$$

where V'_a is the irreducible G-module V_a, considered as an $S(\mathbb{C}^2, G)$-module with the trivial action of $S^{>0}(\mathbb{C}^2, G)$. Kapranov-Vasserot [KV] equivalence of derived categories of $S(\mathbb{C}^2, G)$-modules and coherent sheaves on the minimal resolution of $\mathbb{C}^2/G$ allows us to transfer the braid group action to the latter category. $\square$

Assume that $G \subset SU(2)$ is *binary* and let $G' \cong G/\{\pm 1\}$ be the image of G in $SO(3)$. G' acts on $\mathbb{P}^1$ and this action induces an action on the cotangent bundle $T^*\mathbb{P}^1$. Consider the category $Coh_{G'}(T^*\mathbb{P}^1)$ of G'-equivariant coherent sheaves on $T^*\mathbb{P}^1$ and its derived category $D_{G'}(T^*\mathbb{P}^1)$.

G' acts on $T^*\mathbb{P}^1$ with isolated singular points only. The quotient variety $T^*\mathbb{P}^1/G'$ has two or three singular points (two if G' is cyclic), of type $\mathbb{C}^2/\mathbb{Z}_k$, where $\mathbb{Z}_k \subset SU(2)$ is the stabilizer subgroup of the corresponding point on $\mathbb{P}^1$.

Proposition 19 *The minimal resolution of $T^*\mathbb{P}^1/G'$ is isomorphic to the minimal resolution of $\mathbb{C}^2/G$.*

This is proved in Lamotke [La]. $\square$

Kapranov-Vasserot theorem [KV] implies

Proposition 20 *The categories of G'-equivariant coherent sheaves on $T^*\mathbb{P}^1$ and coherent sheaves on the minimal resolution of simple singularity $\mathbb{C}^2/G$ are derived equivalent.*

What are the multiplicities of various irredicible representations of $G \subset SU(2)$ in the n-th symmetric power $S^n(\mathbb{C}^2)$ of the "basic" representation $\mathbb{C}^2$? This problem was solved in different ways by Gonzalez-Springer and Verdier [G-SV], Knörrer [Kn], Kostant [Ks] and Springer [Sp]. We offer an interpretation via quantum Cartan matrices (Section 6.1) and the algebra $S(\mathbb{C}^2, G)$ as follows.

We restrict to the case of binary G. As before, let a be a vertex of Γ^{aff} and V_a the irreducible representation of G associated to a. Let t be the vertex associated to the trivial representation V_t of G. There is an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathbb{C}[G]}(V_a, S^n(\mathbb{C}^2)) \cong \mathrm{Hom}_{\mathbb{C}[G]^{\otimes 2}}(V_a \otimes V_t, S^n(\mathbb{C}^2) \otimes \mathbb{C}[G]) \qquad (43)$$

where $g \otimes h \in \mathbb{C}[G]^{\otimes 2}$ takes $v_1 \otimes v_2 \in V_a \otimes V_t$ to $gv_1 \otimes v_2$ and $s \otimes f \in S^n(\mathbb{C}^2) \otimes \mathbb{C}[G]$ to $gs \otimes gfh^{-1}$. To $\alpha \in$ L.H.S. the isomorphism associates $\alpha \otimes \beta$, where

$$\beta : V_t \longrightarrow \mathbb{C}[G], \qquad \beta(v_2) = \sum_{g \in G} g.$$

Notice that $S^n(\mathbb{C}^2)\otimes\mathbb{C}[G]$ is just the n-th degree component of the algebra $S(\mathbb{C}^2,G)$, Morita equivalent to $A^!(\Gamma^{\mathrm{aff}})$. We can rewrite the R.H.S. of (43) in terms of $A^!(\Gamma^{\mathrm{aff}})$. Namely, $\mathbb{C}[G]$, the degree 0 component of $S(\mathbb{C}^2,G)$, becomes the degree 0 component of $A^!(\Gamma^{\mathrm{aff}})$, isomorphic to the direct sum of $\mathbb{C}$'s, one for each vertex of the affine diagram. Denote this algebra by A and its simple modules corresponding to a and t by L_a, L_t. Then

$$\mathrm{Hom}_{\mathbb{C}[G]^{\otimes 2}}(V_a\otimes V_t, S(\mathbb{C}^2)\otimes\mathbb{C}[G])\cong \mathrm{Hom}_{A\otimes A^{op}}(L_a\otimes L_t, A^!(\Gamma^{\mathrm{aff}})).$$

Therefore, the right hand side of (43) is isomorphic to the n-th degree component of the vector space $(a)A^!(\Gamma^{\mathrm{aff}})(t)$, where we multiplied $A^!(\Gamma^{\mathrm{aff}})$ on the left, resp. right, by minimal idempotents associated to a, resp. t. The dimension of this vector space is equal to the coefficient of q^n in the power series $c^!_{a,t}$ (see formula (37)). Applying Proposition 12 we obtain

Proposition 21 *If G is binary, the multiplicity of the irreducible representation V_a of G in $S^n(\mathbb{C}^2)$ is equal to $(-1)^n$ times the coefficient at q^n of the (a,t)-entry of the inverse quantum Cartan matrix of the affine Dynkin diagram associated to G via the McKay correspondence.*

Remark The restriction on G is unnecessary. Essentially the same proof, with the skew-zigzag algebra (42) substituted for $A^!(\Gamma^{\mathrm{aff}})$, works when G is non-binary. We leave the details to the reader. Numerically, however, the case of cyclic G is boring, since every irreducible representation is one-dimensional.

We conclude this advertisement of cross-products $S(\mathbb{C}^2,G)$ and $\Lambda(\mathbb{C}^2,G)$ as vital ingredients in the McKay correspondence by referring the reader to Auslander [Au] and Reiten [R] for a relation between $S(\mathbb{C}^2,G)$ and AR quivers of ADE singularities.

7 Zigzag algebras in representation theory and geometry

7.1 Zigzag algebras and finite groups

Until now we worked over complex numbers and $A(\Gamma)$ was defined over $\mathbb{C}$. In fact, $A(\Gamma)$ is defined over the ring of integers, and we denote this $\mathbb{Z}$-algebra by $A_{\mathbb{Z}}(\Gamma)$. For a commutative ring k denote by $A_k(\Gamma)$ the k-algebra $A_{\mathbb{Z}}(\Gamma)\otimes_{\mathbb{Z}} k$. The following examples show that $A_k(\Gamma)$ and Morita equivalent algebras often appear as direct summands of group algebras $k[G]$ in finite characteristic. Let us denote by Γ_n the chain with n vertices:

1. Let p be a prime and k a field of characteristic p.

Proposition 22 *The group algebra $k[\mathbb{S}_p]$ of the symmetric group is the direct sum of a semisimple algebra and an algebra Morita equivalent to $A_k(\Gamma_{p-1})$.*

This is an exercise in the modular representation theory of symmetric groups. □

A similar statement holds for Hecke algebras (see Yamane [Y]):

Proposition 23 *The Hecke algebra $H_q(\mathbb{S}_n)$ when q is an n-th primitive root of unity is isomorphic to the direct sum of a semisimple algebra and an algebra Morita equivalent to $A_{\mathbb{C}}(\Gamma_{n-1})$.*

2. The group algebra of the finite group $SL(2,p)$ over an algebraically closed field k of characteristic $p>2$ decomposes as a direct sum of 3 blocks, one of which is simple and the other two have Cartan matrices

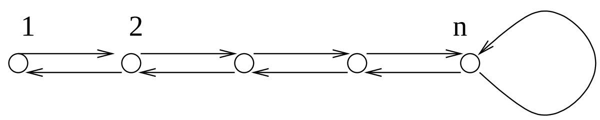
(see Alperin [Al], §17)

$$\begin{pmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & \cdot & & & \\ & & \cdot & \cdot & 1 & & \\ & & & 1 & 2 & 1 & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 3 \end{pmatrix}$$

In each of these two blocks, if we throw away the indecomposable projective module P with $\dim(\mathrm{Hom}(P,P)) = 3$, the endomorphism algebra $\mathrm{End}(P_1 \oplus \cdots \oplus P_{\frac{p-3}{2}})$ of the direct sum of the remaining indecomposable projectives is isomorphic to the algebra $A_k(\Gamma_{\frac{p-3}{2}})$. The presence of the extra projective P does not create any obstacles for defining a braid group action, and in the derived category of this block there is a faithful action of the braid group on $\frac{p-1}{2}$ strands.

3. Let Di_6 be the dihedral group of order 12 and k an algebraically closed field of characteristic 3. Then the group algebra $k[\mathrm{Di}_6]$ is isomorphic to the direct sum $A_k(\Gamma_2) \oplus A_k(\Gamma_2)$. This result can be easily derived from the computation in Curtis and Reiner [CR], §91 of the Cartan matrix of Di_6 in characteristic 3.

4. The following example generalizes **1, 2** and **3**. Let $B(n,r)$ be the quiver algebra



with defining relations

$$(i|i+1|i+2) = (i|i-1|i-2) = (n-1|n|n) = (n|n|n-1) = (n|n)^{r+1} = 0,$$
$$(i|i-1|i) = (i|i+1|i), \quad (n|n-1|n) = (n|n)^r.$$

Algebra $B(n,r)$ is an example of a Brauer tree algebra with $n+1$ vertices and exceptional multiplicity r. Alperin [Al] calls it the *open polygon* Brauer tree algebra (vertices in Brauer trees do not correspond to vertices in the quiver algebra, hence the discrepancy between n and $n+1$). If $r > 1$, we will call the rightmost vertex exceptional.

Brauer tree algebras and Morita equivalent algebras are isomorphic to the so-called cyclic defect blocks of group algebras of finite groups over algebraically closed fields of finite characteristic ([Al],[Fe],[KZ]). This description of cyclic defect blocks was an important achivement of the modular representation theory. Cyclic defect blocks are quite widespread. For example, the classification of all blocks of cyclic defect and their Brauer trees in sporadic simple groups takes up over four hundred pages in the monograph [HL].

Rickard ([Ri1], see also [KZ], chapters 5 and 10) showed that any Brauer tree algebra with $n+1$ vertices and multiplicity r is derived equivalent to $B(n,r)$.

The subalgebra of the algebra $B(n,r)$ consisting of all paths that neither start nor end in the exceptional vertex is isomorphic to the zigzag algebra $A(\Gamma_{n-1})$ (or $A(\Gamma_n)$ if $r=1$).

In particular, the results of [KS] imply that there is a faithful action of the n-stranded braid group in the derived category of $B(n,r)$ and of the $(n+1)$-stranded braid group in the derived category of $B(n,1)$. The braid group generators act by tensoring with the complex of bimodules

$$0 \longrightarrow P_i \otimes {}_iP \longrightarrow B(n,r) \longrightarrow 0,$$

where P_i, respectively ${}_iP$, is the left, resp. right, indecomposable projective for the vertex i.

When this paper was ready for publication we learned that this braid group action was independently discovered by A.Zimmermann and R.Rouquier in [RZ]. They proved that the braid group action is faithful for the 3-stranded braid group. In addition, the survey paper of Rouquier [Ro] offers several startling conjectures about braid group actions in derived categories.

5. Let Γ be the cyclic graph with 3 vertices and 3 edges and k a field of characteristic 2 that contains a cubic root of 1. The algebra $A_k(\Gamma)$ is isomorphic to the group algebra (over k) of the alternating group $\mathbb{A}_4$ (see Erdmann [E], page 62). More examples of blocks Morita equivalent and derived Morita equivalent to $A_k(\Gamma)$ can be extracted from Section 12.6 of Feit [Fe] (and see Rickard [Ri2] for a derived equivalence between the group algebra of $\mathbb{A}_4$ and the principal block of $\mathbb{A}_5$ in characteristic 2).

7.2 Zigzag algebras and the Lie algebra $\mathfrak{sl}_2$

Let V be the fundamental representation of the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{C}$. Denote by $\mathfrak{sl}_2(V)$ the 5-dimensional Lie algebra isomorphic as a vector space to $\mathfrak{sl}_2 \oplus V$ with the Lie bracket

$$[(x,a),(y,b)] = ([x,y], xb - ya), \quad \text{where} \quad x,y \in \mathfrak{sl}_2,\ a,b \in V. \tag{44}$$

The following observation is due to Loupias [Lp]

Proposition 24 *The category of finite-dimensional $\mathfrak{sl}_2(V)$ representations is equivalent to the category of finite-dimensional modules over the quiver algebra*

$$\overset{0}{\circ} \rightleftarrows \overset{1}{\circ} \rightleftarrows \overset{2}{\circ} \rightleftarrows \dots \rightleftarrows \overset{i-1}{\circ} \rightleftarrows \overset{i}{\circ} \rightleftarrows \dots \tag{45}$$

with relations $(0|1|0) = 0$ *and* $(i|i+1|i) = (i|i-1|i)$.

Denote by $\mathfrak{sl}_2^-(V)$ the $(3|2)$-dimensional super Lie algebra isomorphic as a super vector space to $(\mathfrak{sl}_2, V)$ with the super Lie bracket

$$[(x,a),(y,b)] = ([x,y], xb + ya), \quad \text{where} \quad x,y \in \mathfrak{sl}_2,\ a,b \in V. \tag{46}$$

Proposition 25 *The category of finite-dimensional $\mathfrak{sl}_2^-(V)$ representations is equivalent to the category of finite-dimensional modules over the quiver algebra*

$$\overset{0}{\circ} \rightleftarrows \overset{1}{\circ} \rightleftarrows \overset{2}{\circ} \rightleftarrows \dots \rightleftarrows \overset{i-1}{\circ} \rightleftarrows \overset{i}{\circ} \rightleftarrows \dots \tag{47}$$

with relations $(i|i+1|i+2) = 0, (i|i-1|i-2) = 0$ *and* $(i|i+1|i) = (i|i-1|i)$.

Note that algebras described in propositions 24 and 25 are quadratic and in fact quadratic dual to each other and Koszul. The second algebra is isomorphic to the zigzag algebra of the infinite in one direction chain:

$$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\cdots\cdots$$

This duality between representations of $\mathfrak{sl}_2(V)$ and $\mathfrak{sl}_2^-(V)$ can be generalized to arbitrary pairs $(\mathfrak{g}, V)$ where $\mathfrak{g}$ is a semisimple Lie algebra and V a finite-dimensional representation of $\mathfrak{g}$. One can form the Lie algebra $\mathfrak{g}(V) = \mathfrak{g} \oplus V$ with the bracket (44) and the "dual" Lie superalgebra $\mathfrak{g}^-(V^*)$. The categories of finite-dimensional representations of $\mathfrak{g}(V)$ and $\mathfrak{g}^-(V^*)$ are described by Koszul dual algebras.

7.3 Other examples

1. Let T be an elliptic curve and p a point of T. Let $\mathcal{L}$ be the direct sum of the structure sheaf $\mathcal{O}_T$ and the skyscraper sheaf $\mathbb{C}_p$. The ext algebra $\mathrm{Ext}_{\mathrm{Coh}(T)}(\mathcal{L},\mathcal{L})$ is isomorphic to $A(\Gamma_2)$. Seidel and Thomas [ST] and Thomas [T] list many other appearances of algebras $A(\Gamma)$ as ext algebras of sheaves on Calabi-Yau varieties.

2. Algebra $A(\Gamma_n)$ is isomorphic to the algebra of Floer homology groups $\bigoplus_{1\le i,j\le n} HF(L_i, L_j)$ where $L_1, \dots, L_n$ is a chain of Lagrangian spheres in a suitable symplectic manifold (see [KS]).

3. Let $PS(\mathbb{P}^n)$ be the category of perverse sheaves on $\mathbb{P}^n$, smooth along the stratification of $\mathbb{P}^n$ by the increasing chain $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$. Let Q_i be the indecomposable projective perverse sheaf associated to the i-dimensional strata and $Q = \underset{0 \le i \le n-1}{\oplus} Q_i$. The zigzag algebra $A(\Gamma_n)$ is isomorphic to the endomorphism algebra $\mathrm{End}_{PS(\mathbb{P}^n)}(Q)$ (see [KS] for details).

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